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LETTER TO THE EDITOR

Axial anomaly and index theorem for a two-dimensional disc with boundary†

Zhong-Qi Ma‡

Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305, USA

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Abstract. As a complement of the paper of Ninomiya and Tan, we obtain a correct index formula for a two-dimensional disc with a boundary which is a special case of the APS theorem.

Recently, we read a beautiful paper by Ninomiya and Tan (1985) (NT) who discussed the index theorem, as a special case of a general index theorem proved by Atiyah *et al* (1975a, b, 1976) (APS), for the manifold with a boundary in terms of 'handed' non-local boundary conditions. For a manifold with boundaries NT obtained the general expression of the index theorem, including the boundary contributions:

$$n_+ - n_- = \int_X A(X)\,\mathrm{d}X - \sum_j f(Y_j) \tag{1}$$

where $A(X)$ is the anomaly density, $n_+(n_-)$ is the number of chirality positive (negative) zero energy solutions to the Dirac eigenvalue equation and $f(Y_j)$ is the surface contribution of the jth boundary.

In physics, we usually use two kinds of coordinates: those with length dimension and the angular coordinates. There are some characteristics for the angular coordinates which should be dealt with carefully. Unfortunately, NT made some mistakes in the index theorem for the angular coordinates. In order to make the problem clear, we only discuss the index theorem for a two-dimensional disc with a boundary in this letter. We use the same notation as NT.

In a two-dimensional manifold we introduce a circular boundary about an appropriate origin, i.e. restricting $0 \leqslant r \leqslant \rho$ in polar coordinates (r, θ). Adopting a 'radial' gauge where the general gauge potential is along the $\hat{\theta}$ direction,

$$V = \hat{\theta} V(r, \theta) = (-\sin\theta\hat{x} + \cos\theta\hat{y})\, V(r, \theta). \tag{2}$$

The massless Dirac operator is

$$\mathrm{i}\not{D} = -\mathrm{i}\sigma_2(\partial_x - \mathrm{i}V_x) + \mathrm{i}\sigma_1(\partial_y - \mathrm{i}V_y) = \begin{pmatrix} & L^+ \\ L & \end{pmatrix} \tag{3}$$

$$L = \mathrm{e}^{\mathrm{i}\theta}[\partial_r + B_r(\theta)] \qquad L^+ = \mathrm{e}^{-\mathrm{i}\theta}[-\partial_r + B_r(\theta)] \tag{4}$$

$$B_r(\theta) = (\mathrm{i}/r)\,\partial_\theta + V(r, \theta) \tag{5}$$

where $V(r, \theta)$ is real, and $B_r(\theta)$ is, for a fixed r, self-adjoint over the periodic interval.

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‡ On leave from the Institute of High Energy Physics, Academia Sinica, PO Box 918, Beijing, People's Republic of China.

The eigenfunction of $i\not{D}$ is denoted by ψ_E:

$$i\not{D}\psi_E = E\psi_E \qquad \psi_E = \begin{bmatrix} \phi_E(r, \theta) \\ \chi_E(r, \theta) \end{bmatrix} \tag{6}$$

$$L\phi_E = E\chi_E \qquad L^+\chi_E = E\phi_E. \tag{7}$$

As a first step towards deriving an index theorem, proper boundary conditions must be imposed so that L and L^+ are true adjoint of each other, that is, $\langle\phi|L^+|\chi\rangle = \langle\chi|L|\phi\rangle^*$. However, owing to the factor $e^{i\theta}(e^{-i\theta})$ in $L(L^+)$, the global boundary condition has a different form from equation (3.2) in Atiyah *et al* (1975a). From (4) we have

$$\begin{aligned} L &= (e^{i\theta/2}/\sqrt{r})[\partial_r + B_r(\theta)]\, e^{i\theta/2}\sqrt{r} \\ L^+ &= (e^{-i\theta/2}/\sqrt{r})[-\partial_r + B_r(\theta)]\, e^{-i\theta/2}\sqrt{r}. \end{aligned} \tag{4$'$}$$

The two components in ψ_E have different θ dependences:

$$\psi_E = \frac{1}{\sqrt{r}}\begin{bmatrix} e^{-i\theta/2} & \phi_{1E}(r, \theta) \\ e^{i\theta/2} & \chi_{1E}(r, \theta) \end{bmatrix}. \tag{6$'$}$$

Therefore, we obtain the eigenvalue equations

$$L_1 = \partial_r + B_r(\theta) \qquad L_1^+ = -\partial_r + B_r(\theta) \tag{8}$$

$$L_1\phi_{1E} = E\chi_{1E} \qquad L_1^+\chi_{1E} = E\phi_{1E} \tag{7$'$}$$

and the global boundary condition

$$\int_0^{2\pi} r\, d\theta\, e^{i\theta}\chi_{E'}^*(r, \theta)\phi_E(r, \theta) = \int_0^{2\pi} d\theta\, \chi_{1E'}^*(r, \theta)\phi_{1E}(r, \theta) = 0 \qquad r = 0 \text{ or } \rho. \tag{9}$$

We define the continuous effective potential

$$\bar{V}(r) = \frac{1}{2\pi}\int_0^{2\pi} V(r, \theta)\, d\theta. \tag{10}$$

The anomaly is

$$A = \frac{1}{2\pi}\iint Fr\, dr\, d\theta = \frac{1}{2\pi}\iint \left[\frac{1}{r}\frac{\partial}{\partial r} rV(r, \theta)\right] r\, dr\, d\theta = \rho\bar{V}(\rho) - r\bar{V}(r)|_{r=0}. \tag{11}$$

Usually we demand $\bar{V}(0) = 0$ in order to make gauge potential V single-valued at the origin. Only for the manifold with a hole around the origin, the non-vanishing $\bar{V}(\delta)$ is allowed, where δ is the radius of the hole, and the anomaly is

$$A = \rho\bar{V}(\rho) - \delta\bar{V}(\delta). \tag{11$'$}$$

Solving the eigenvalue problem of $B_r(\theta)$, we obtain

$$B_r(\theta)e_\lambda(r, \theta) = \lambda_r e_\lambda(r, \theta) \tag{12}$$

$$e_\lambda(r, \theta) = \exp\left(iJ\theta + ir\int_0^\theta [V(r, \theta') - \bar{V}(r)]\, d\theta'\right) \tag{13}$$

$$\lambda_r = -(Jr^{-1}) + \bar{V}(r) \tag{14}$$

where J is half of the odd integer so that ψ_E is single-valued. Now, at the boundaries,

$r=\delta$ and ρ, we expand $\phi_{1E}(r,\theta)$ and $\chi_{1E}(r,\theta)$

$$\left.\begin{aligned}\phi_{1E}(r,\theta)&=\sum_\lambda f_{\lambda E}(r)e_\lambda(r,\theta)\\ \chi_{1E}(r,\theta)&=\sum_\lambda g_{\lambda E}(r)e_\lambda(r,\theta)\end{aligned}\right\}\qquad r=\delta \text{ and } \rho \tag{15}$$

and impose the 'left-handed' global boundary condition at $r=\delta$ and the 'right-handed' one at $r=\rho$†

$$\begin{aligned}
&f_{\lambda E}(\delta)=0 &&\text{for } \lambda_\delta\geqslant 0 &&f'_{\lambda E}(\delta)/f_{\lambda E}(\delta)=-\lambda_\delta &&\text{for } \lambda_\delta<0\\
&g_{\lambda E}(\delta)=0 &&\text{for } \lambda_\delta< 0 &&g'_{\lambda E}(\delta)/g_{\lambda E}(\delta)=\lambda_\delta &&\text{for } \lambda_\delta\geqslant 0 \qquad (16a)\\
&f_{\lambda E}(\rho)=0 &&\text{for } \lambda_\rho< 0 &&f'_{\lambda E}(\rho)/f_{\lambda E}(\rho)=-\lambda_\rho &&\text{for } \lambda_\rho\geqslant 0\\
&g_{\lambda E}(\rho)=0 &&\text{for } \lambda_\rho\geqslant 0 &&g'_{\lambda E}(\rho)/g_{\lambda E}(\rho)=\lambda_\rho &&\text{for } \lambda_\rho<0. \qquad (16b)
\end{aligned}$$

With these conditions, L_1 and L_1^+ are adjoints of each other and the APS theorem (1) can in principle be applied. The boundary contributions are

$$f(\rho)=\tfrac{1}{2}[\eta(\rho)-h_\rho]\qquad f(\delta)=\tfrac{1}{2}[\eta(\delta)+h_\delta] \tag{17}$$

$$\eta(\rho)=-\lim_{s\to\infty}\sum_{\lambda_\rho\neq 0}[\operatorname{sgn}\lambda_\rho]|\lambda_\rho|^{-s}\qquad \eta(\delta)=\lim_{s\to\infty}\sum_{\lambda_\delta\neq 0}[\operatorname{sgn}\lambda_\delta]|\lambda_\delta|^{-s}$$

$$h_r=\dim\ker B_r. \tag{18}$$

Noticing that J is half of the odd integer, we obtain

$$f(\rho)=\langle\rho\bar V(\rho)+\tfrac12\rangle-\tfrac12\qquad f(\delta)=\tfrac12-\langle\delta\bar V(\delta)+\tfrac12\rangle \tag{19}$$

where

$$\langle A\rangle=A-[A] \tag{20}$$

and $[A]$ denotes the largest integer less than or equal to A.

Substituting (11′) and (19) into (1) we find the index of the massless Dirac operator $\mathrm{i}\not{D}$

$$n_+-n_-=A-f(\rho)-f(\delta) \tag{1′}$$

$$\begin{aligned}&=\rho\bar V(\rho)-\delta\bar V(\delta)-\langle\rho\bar V(\rho)+\tfrac12\rangle+\langle\delta\bar V(\delta)+\tfrac12\rangle\\ &=[\rho\bar V(\rho)+\tfrac12]-[\delta\bar V(\delta)+\tfrac12].\end{aligned} \tag{21}$$

For the disc without a hole around the origin

$$\left.\begin{aligned}f(\rho)&=\langle\rho\bar V(\rho)+\tfrac12\rangle-\tfrac12\\ f(0)&=0\end{aligned}\right. \tag{19′}$$

$$\begin{aligned}n_+-n_-&=\rho\bar V(\rho)-\langle\rho\bar V(\rho)+\tfrac12\rangle+\tfrac12\\ &=[\rho\bar V(\rho)+\tfrac12].\end{aligned} \tag{21′}$$

† If the global boundary condition at $r=\rho$ of NT is imposed (see (6.5)–(6.8) in their paper, and $f_\lambda(\rho)\neq 0$ when $\lambda\geqslant 0$, $g_\lambda(\rho)\neq 0$ when $\lambda<0$), the non-vanishing $f_{\lambda_1}(\rho)e_{\lambda_1}(\theta)$ with the lowest $\lambda=\lambda_1$ and the non-vanishing $g_{\lambda_2}(\rho)e_{\lambda_2}(\theta)$ with the largest $\lambda=\lambda_2$ will not satisfy (9), because $\mathrm{e}^{\mathrm{i}\theta}e_{\lambda_1}=e_{\lambda_2}$.

As a check, we discuss a symmetric gauge with $V(r, \theta) = V(r)$ in a disc, $0 \leqslant r \leqslant \rho$, without a hole around the origin. For this case we have

$$\psi_{\mathrm{E}}(r, \theta) = \frac{1}{r}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\theta/2} & f_{\lambda\mathrm{E}}(r) \\ \mathrm{e}^{\mathrm{i}\theta/2} & g_{\lambda\mathrm{E}}(r) \end{bmatrix} e_\lambda(\theta)$$
$$e_\lambda(\theta) = \exp\{\mathrm{i}J\theta\} \qquad \lambda_r = -Jr^{-1} + V(r) \tag{22}$$

$$L_1 = \partial_r - Jr^{-1} + V(r) \qquad L_1^+ = -\partial_r - Jr^{-1} + V(r). \tag{23}$$

The chirality positive and negative zero energy solutions to the Dirac eigenvalue equation are the following:

$$f_{\lambda 0}(r) = \exp\left[\int_1^r \left(\frac{J}{r'} - V(r')\right) \mathrm{d}r'\right]$$
$$\sim \begin{cases} cr^J & r \sim 0 \\ \exp\left(J \ln \rho - \int_1^\rho V(r')\,\mathrm{d}r'\right) & r = \rho \end{cases} \tag{23a}$$

$$g_{\lambda 0}(r) = \exp\left[-\int_1^r \left(\frac{J}{r'} - V(r')\right) \mathrm{d}r'\right]$$
$$\sim \begin{cases} cr^{-J} & r \sim 0 \\ \exp\left(-J \ln \rho + \int_1^\rho V(r')\,\mathrm{d}r'\right) & r = \rho. \end{cases} \tag{23b}$$

$$f'_{\lambda 0}(\rho)/f_{\lambda 0}(\rho) = \rho^{-1}[J - \rho V(\rho)] = -\lambda_\rho$$
$$g'_{\lambda 0}(\rho)/g_{\lambda 0}(\rho) = \rho^{-1}[-J + \rho V(\rho)] = \lambda_\rho \tag{24}$$

where we assume that $\rho > 1$ without loss of generality. Imposing the global boundary conditions (16) at $r = 0$ and $r = \rho$, we obtain (J is half of the odd integer!)

$$\tfrac{1}{2} \leqslant J \leqslant \rho V(\rho) \qquad \text{for } f_{\lambda 0}$$
$$-\tfrac{1}{2} \geqslant J > \rho V(\rho) \qquad \text{for } g_{\lambda 0} \tag{25}$$

and

$$n_+ = [\rho V(\rho) + \tfrac{1}{2}]\theta[V(\rho)]$$
$$n_- = -[\rho V(\rho) + \tfrac{1}{2}]\theta[-V(\rho)] \tag{26}$$

so (21′) follows.

If, for example,

$$V(r) \sim \begin{cases} 0 & r \sim 0 \\ F/r & r \leqslant \rho \end{cases} \tag{27}$$

we have

$$n_+ - n_- = (F + \tfrac{1}{2}). \tag{28}$$

Boyanovsky and Blankenbecler (1985)† also obtained this result for $F + \frac{1}{2} \neq$ integer.

† They imbedded the radial half-line problem onto the full line and recovered the problem of interest by a simple and tame limiting procedure, instead of the global boundary condition (16). See equations (2.54) and (2.76) in their paper.

For the constant H-field example,

$$V(r)=\tfrac{1}{2}Hr \tag{29}$$

we have

$$\phi_0(r,\theta)=r^{J-(1/2)}\exp[\mathrm{i}(J-\tfrac{1}{2})\theta-\tfrac{1}{4}Hr^2] \tag{30a}$$

$$\chi_0(r,\theta)=r^{-J-(1/2)}\exp[\mathrm{i}(J+\tfrac{1}{2})\theta+\tfrac{1}{4}Hr^2]. \tag{30b}$$

Due to the left-handed condition at $r=0$, only $J\geqslant\frac{1}{2}$, i.e. $J-\frac{1}{2}\geqslant 0$, is accepted for (30a), and $J\leqslant-\frac{1}{2}$, i.e. $J+\frac{1}{2}\leqslant 0$, for (30b). Equation (16b) at $r=\rho$ leads to the result that $\frac{1}{2}\leqslant J\leqslant\frac{1}{2}H\rho^2$ for ϕ_0 and $-\frac{1}{2}\geqslant J>\frac{1}{2}H\rho^2$ for χ_0. Together we obtain a compact expression for the index

$$n_+-n_-=[\tfrac{1}{2}H\rho^2+\tfrac{1}{2}]. \tag{31}$$

For the RHS of the APS theorem (1′), the first term contributes $\frac{1}{2}H\rho^2$, and the surface term, from (19′), is

$$\begin{aligned} f(\rho)&=\langle\tfrac{1}{2}H\rho^2+\tfrac{1}{2}\rangle-\tfrac{1}{2}\\ f(0)&=0 \end{aligned} \tag{32}$$

which is precisely what is needed for the APS theorem (1′). This result holds whether or not there is a small hole around the origin, because the boundary conditions (9) and (16a) at $r=0$ are satisfied even though $J=\pm\frac{1}{2}$. If there is a small hole with a radius δ around the origin and $V(\delta)\neq 0$, we have

$$\begin{aligned} f(\rho)&=\langle\tfrac{1}{2}H\rho^2+\tfrac{1}{2}\rangle-\tfrac{1}{2}\\ f(\delta)&=\tfrac{1}{2}-\langle\tfrac{1}{2}H\delta^2+\tfrac{1}{2}\rangle \end{aligned} \tag{33}$$

and $\frac{1}{2}H\delta^2<J\leqslant\frac{1}{2}H\rho^2$ for ϕ_0 and $-\frac{1}{2}H\delta^2\geqslant J>\frac{1}{2}H\rho^2$ for χ_0,

$$\begin{aligned} n_+-n_-&=[\tfrac{1}{2}H\rho^2+\tfrac{1}{2}]-[\tfrac{1}{2}H\delta^2+\tfrac{1}{2}]\\ &=\tfrac{1}{2}H(\rho^2-\delta^2)-f(\rho)-f(\delta). \end{aligned} \tag{34}$$

In summary, Ninomiya and Tan wrote a beautiful paper; unfortunately, with a defect. This letter is only a complement to their paper.

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